

# A priori estimates for excitable models

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## Abstract

The reaction-diffusion system of Fitzhugh Nagumo is considered. The initial-boundary problems with Neumann and Dirichlet conditions are analyzed. By means of an equivalent semilinear integrodifferential equation which characterizes several dissipative models of viscoelasticity, biology, and superconductivity, some results on existence, uniqueness and a priori estimates are deduced both in the linear case and in the non linear one.

**Keywords:** Reaction - diffusion systems; Biological applications; Laplace transform, FitzHugh Nagumo model.

**Mathematics Subject Classification (2000)** 35E05 35K35 35K57 35Q53 78A70

## 1 Introduction

As it is well known, the FitzHugh - Nagumo model (FHN) has been introduced by FitzHugh (1961) and Nagumo et al. (1962) as a simplified description of the excitation and propagation of nerve impulses [1, 2].

Of course, there are many other important physiological applications in addition to that for the propagation of nerve action potentials modeled by the FHN system. One such important application is related to the waves that arise in muscle tissue, particularly heart muscle and particularly interesting is the biophysical phenomenon of reentry which occurs in the excitable cells of the heart. [3] Another example is the reverberating cortical depression waves in the brain cortex. [4] As for the analysis of the (FHN) model, there exists a large bibliography both in the linear case and non linear problem. [5, 7–9].

The model is given by the set of p.d.e [4, 10]:

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$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - v + f(u) \\ \frac{\partial v}{\partial t} = b u - \beta v. \end{cases}$$

where  $u(x, t)$  models the transmembrane voltage of a nerve axon at distance  $x$  and time  $t$ ,  $v(x, t)$  is an auxiliary variable that acts as recovery variable. Further, the diffusion coefficient  $\varepsilon$  and the parameters  $b, \beta$  are all non negative [11, 12].

Besides, the function  $f(u)$  has the qualitative form of a cubic polynomial

$$(1.2) \quad f(u) = -a u + \varphi(u) \quad \text{with} \quad \varphi = u^2(a + 1 - u)$$

where the parameter  $a$  is the threshold constant and generally one has  $0 < a < 1$ . However, in many papers phenomena for the (FHN) system has been investigated considering  $a$  as a non homogeneous function and  $a < 0$ . [13, 14]

Moreover, Denoting by  $v_0$  the initial value of  $v$ , system (1.1) can be given the form of the following integro differential equation

$$(1.3) \quad \mathcal{L} u \equiv u_t - \varepsilon u_{xx} + a u + b \int_0^t e^{-\beta(t-\tau)} u(x, \tau) d\tau = F(x, t, u(x, t))$$

as soon as one puts:

$$(1.4) \quad F(x, t, u) = \varphi(u) - v_0(x) e^{-\beta t}.$$

Equation (1.3) describes the evolution of several physical models as motions of viscoelastic fluids or solids [15–17]; heat conduction at low temperature [18], sound propagation in viscous gases [19].

Moreover, it occurs also in superconductivity to describe the Josephson tunnel effects in junctions. In this case the unknown  $u$  denotes the difference between the phases of the wave functions of the two superconductors and it results:

$$(1.5) \quad \varepsilon u_{xxt} - u_{tt} + u_{xx} - \alpha u_t = \sin u + \gamma$$

where  $\gamma$  is a constant forcing term that is proportional to a bias current, and the  $\varepsilon$  – term and the  $\alpha$  – term account for the dissipative normal electron current flow along and across the junction, respectively [20].

From (1.3) one obtains the equation (1.5) as soon as one assumes

$$(1.6) \quad a = \alpha - \frac{1}{\varepsilon} \quad b = -\frac{a}{\varepsilon} \quad \beta = \frac{1}{\varepsilon}$$

and  $F$  is such that

$$(1.7) \quad F(x, t, u) = - \int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)} [\gamma + \text{sen } u(x, \tau)] d\tau.$$

As (1.6) show, in the superconductive case the constants  $a, b$  could be negative too.

In this paper Neumann and the Dirichlet problem for (1.3) are considered. By means of the fundamental solution  $K_0(x, t)$  of the operator  $L$ , which has already been determined in [21], in the linear case, the explicit solutions are obtained. When the source term  $F$  is non linear, an appropriate analysis of the integro differential equation implies results on the existence and uniqueness of solutions. Besides, in both the linear case and the non linear problem, a priori estimates of the solutions are achieved. The results are applied to (FHN) system.

## 2 Statement of the problems and Laplace transform

If  $T$  is an arbitrary positive constant and

$$\Omega_T \equiv \{ (x, t) : 0 \leq x \leq L ; 0 < t \leq T \},$$

let  $(P_N)$  the following Neumann initial - boundary value problem related to equation (1.3):

$$(2.1) \quad \begin{cases} u_t - \varepsilon u_{xx} + au + b \int_0^t e^{-\beta(t-\tau)} u(x, \tau) d\tau = F(x, t, u) & (x, t) \in \Omega_T \\ u(x, 0) = u_0(x) & x \in [0, L], \\ u_x(0, t) = \psi_1(t) \quad u_x(L, t) = \psi_2(t) & 0 < t \leq T. \end{cases}$$

In excitable systems this problem occurs when two-species reaction diffusion system is subjected to flux boundary condition [4]. The same conditions are present in case of pacemakers [2]. Neumann conditions are applied also to study distributed FHN system [22].

Besides, if the Dirichlet conditions are applied, the following initial boundary value problem  $(P_D)$  holds:

$$(2.2) \quad \begin{cases} \mathcal{L} u = F(x, t, u) & (x, t) \in \Omega_T \\ u(x, 0) = u_0(x) & x \in [0, L], \\ u(0, t) = g_1(t) \quad u(L, t) = g_2(t) & 0 < t \leq T. \end{cases}$$

For example, in mathematical biology, those boundary conditions occur when the behavior of a single dendrite has to determine and the voltage level has to be fixed. [2]

When  $F = f(x, t)$  is a linear function, problems  $(P_N), (P_D)$  can be solved by Laplace transform with respect to  $t$ .

If

$$(2.3) \quad \begin{cases} \hat{u}(x, s) = \int_0^\infty e^{-st} u(x, t) dt & \hat{f}(x, s) = \int_0^\infty e^{-st} f(x, t) dt, \\ \hat{\psi}_i(s) = \int_0^\infty e^{-st} \psi_i(t) dt & (i = 1, 2), \end{cases}$$

one deduces the following transform  $(\hat{P}_N)$  problem:

$$(2.4) \quad \begin{cases} \hat{u}_{xx} - \frac{\sigma^2}{\varepsilon} \hat{u} = -\frac{1}{\varepsilon} [\hat{f}(x, s) + u_0(x)] \\ \hat{u}_x(0, s) = \hat{\psi}_1(s) \quad \hat{u}_x(L, s) = \hat{\psi}_2(s), \end{cases}$$

where  $\sigma^2 = s + a + \frac{b}{s + \beta}$ . Letting  $\tilde{\sigma}^2 = \sigma^2/\varepsilon$ , and considering the following function

$$(2.5) \quad \begin{aligned} \hat{\theta}_0(y, \tilde{\sigma}) &= \frac{\cosh[\tilde{\sigma}(L - y)]}{2\varepsilon\tilde{\sigma}\sinh(\tilde{\sigma}L)} = \\ &= \frac{1}{2\sqrt{\varepsilon}\sigma} \left\{ e^{-\frac{y}{\sqrt{\varepsilon}}\sigma} + \sum_{n=1}^{\infty} \left[ e^{-\frac{2nL+y}{\sqrt{\varepsilon}}\sigma} + e^{-\frac{2nL-y}{\sqrt{\varepsilon}}\sigma} \right] \right\}, \end{aligned}$$

the formal solutions  $\hat{u}(x, s)$  of the problems  $(\hat{P}_N)$  can be given the form:

$$(2.6) \quad \begin{aligned} \hat{u}(x, s) &= \int_0^L [\hat{\theta}_0(|x - \xi|, s) + \hat{\theta}_0(|x + \xi|, s)] [u_0(\xi) + \hat{f}(\xi, s)] d\xi - \\ &- 2\varepsilon\hat{\psi}_1(s)\hat{\theta}_0(x, s) + 2\varepsilon\hat{\psi}_2(s)\hat{\theta}_0(L - x, s). \end{aligned}$$

Analogously, for problem  $(\hat{P}_D)$ , one obtains:

$$(2.7) \quad \begin{aligned} \hat{u}(x, s) = & \int_0^L [\hat{\theta}_0(x + \xi, s) - \hat{\theta}_0(|x - \xi|, s)] [u_0(\xi) + \hat{f}(\xi, s)] d\xi - \\ & - 2\varepsilon \hat{g}_1(s) \frac{\partial}{\partial x} \hat{\theta}_0(x, s) + 2\varepsilon \hat{\psi}_2(s) \frac{\partial}{\partial x} \hat{\theta}_0(L - x, s). \end{aligned}$$

### 3 Fundamental solution $K_0(x, t)$ and theta function $\theta_0(x, t)$

The fundamental solution  $K_0(x, t)$  of the linear operator  $\mathcal{L}$  defined in (1.3) has been already obtained in [21] and it results:

$$(3.1) \quad K_0(r, t) = \frac{1}{2\sqrt{\pi\varepsilon}} \left[ \frac{e^{-\frac{r^2}{4t} - at}}{\sqrt{t}} - \sqrt{b} \int_0^t \frac{e^{-\frac{r^2}{4y} - ay}}{\sqrt{t-y}} e^{-\beta(t-y)} J_1(2\sqrt{by(t-y)}) dy \right],$$

where  $r = |x|/\sqrt{\varepsilon}$  and  $J_n(z)$  is the Bessel function of first kind with

$$(3.2) \quad \mathcal{L}_t K_0 \equiv \int_0^\infty e^{-st} K_0(r, t) dt = \frac{e^{-r\sigma}}{2\sqrt{\varepsilon}\sigma} \quad \Re s > \max(-a, -\beta).$$

Indicating by  $\omega = \min(a, \beta)$  and

$$(3.3) \quad E(t) = \frac{e^{-\beta t} - e^{-at}}{a - \beta} > 0, \quad \beta_0 = \frac{1}{a} + \pi\sqrt{b} \frac{a + \beta}{2(a\beta)^{3/2}},$$

it results [21]:

$$(3.4) \quad |K_0| \leq \frac{e^{-\frac{r^2}{4t}}}{2\sqrt{\pi\varepsilon t}} [e^{-at} + bt E(t)]; \quad \int_0^t d\tau \int_{\mathbb{R}} |K_0(x - \xi, t)| d\xi \leq \beta_0,$$

$$(3.5) \quad \int_{\mathbb{R}} |K_0(x - \xi, t)| d\xi \leq e^{-at} + \sqrt{b}\pi t e^{-\omega t}.$$

In order to obtain the inverse formulae for (2.7), let apply (3.2) to (2.5). Then one deduces the following function similar to *theta functions*:

$$\begin{aligned}
(3.6) \quad \theta_0(x, t) &= K_0(x, t) + \sum_{n=1}^{\infty} [K_0(x + 2nL, t) + K_0(x - 2nL, t)] = \\
&= \sum_{n=-\infty}^{\infty} K_0(x + 2nL, t).
\end{aligned}$$

As consequence, by (2.7), the explicit solution of the *linear* problem  $(P_N)$  where  $F = f(x, t)$  is :

$$\begin{aligned}
(3.7) \quad u(x, t) &= \int_0^L [\theta_0(|x - \xi|, t) + \theta_0(x + \xi, t)] u_0(\xi) d\xi + \\
&- 2\varepsilon \int_0^t \theta_0(x, t - \tau) \psi_1(\tau) d\tau + 2\varepsilon \int_0^t \theta_0(L - x, t - \tau) \psi_2(\tau) d\tau \\
&+ \int_0^t d\tau \int_0^L [\theta_0(|x - \xi|, t - \tau) + \theta_0(x + \xi, t - \tau)] f(\xi, \tau) d\xi.
\end{aligned}$$

In an analogous way, a similar formula for the problem  $(P_D)$  can be obtained:

$$\begin{aligned}
(3.8) \quad u(x, t) &= \int_0^L [\theta_0(x + \xi, t) - \theta_0(|x - \xi|, t)] u_0(\xi) d\xi + \\
&- 2\varepsilon \int_0^t \frac{\partial}{\partial x} \theta_0(x, t - \tau) g_1(\tau) d\tau + 2\varepsilon \int_0^t \frac{\partial}{\partial x} \theta_0(L - x, t - \tau) g_2(\tau) d\tau \\
&+ \int_0^t d\tau \int_0^L [\theta_0(x + \xi, t - \tau) - \theta_0(|x - \xi|, t - \tau)] f(\xi, \tau) d\xi.
\end{aligned}$$

Owing to the basic properties of  $K_0(x, t)$ , it is easy to deduce the following theorems:

**Teorema 3.1.** *When the linear source  $f(x, t)$  is continuous in  $\Omega_T$  and the initial boundary data  $u_0(x)$ ,  $\psi_i(t)$  ( $i = 1, 2$ ) [ $g_i$  ( $i = 1, 2$ )] are continuous, then problem  $(P_N)$  [ $(P_D)$ ] admits a unique regular solution  $u(x, t)$  given by (3.7) [(3.8)].* ■

As consequence of the properties of fundamental solution  $K_0(x, t)$ , various estimates for  $u$ ,  $u_t$ ,  $u_x$ ... could be obtained.

As an example, let evaluate the asymptotic properties of the terms caused by the initial datum  $u_0(x)$  and the source  $f(x, t)$ . If

$$\|u_0\| = \sup_{0 \leq x \leq L} |u_0(x)|, \quad \|f\| = \sup_{\Omega_T} |f(x, t)|,$$

it results:

**Teorema 3.2.** *When  $\psi_i = 0$  ( $i = 1, 2$ ),  $[g_i = 0$  ( $i = 1, 2$ )] the solution (3.7) [(3.8)] of  $(P_N)$ ,  $[(P_D)]$  for large  $t$ , verifies the following estimate:*

$$(3.9) \quad |u(x, t)| \leq 2 \left[ \|f\| \beta_0 + \|u_0\| (1 + \sqrt{b} \pi t) e^{-\omega t} \right]$$

where  $\omega = \min(a, \beta)$  and  $\beta_0$  is given by  $\beta_0 = \frac{1}{a} + \pi \sqrt{b} \frac{a + \beta}{2(a\beta)^{3/2}}$

Proof: Properties of  $K_0(x, t)$  imply that:

$$(3.10) \quad \left| \int_0^L \theta_0(|x - \xi|, t) d\xi \right| \leq \sum_{n=-\infty}^{\infty} \int_0^L |K_0(|x - \xi + 2nL|, t)| d\xi =$$

$$= \sum_{n=-\infty}^{\infty} \int_{x+(2n-1)L}^{x+2nL} |K_0(y, t)| dy \leq \int_{\mathbb{R}} |K_0(y, t)| dy.$$

So, applying properties (3.4)<sub>2</sub> and (3.5) to (3.7) [(3.8)], the estimate (3.9) follows. ■

## 4 The FitzHugh-Nagumo model. A priori estimates

*Linear case* - If the reaction kinetics of the model can be outlined by means of piecewise linear approximations, then one has:

$$(4.1) \quad f(u) = \eta(u - a) - u \quad (0 < a < 1)$$

where  $\eta$  denotes the unit- step function [23, 24]. Like in (1.2), the linear approximation (4.1) involve a linear term  $-u$ . As consequence, denoting by  $\bar{\eta}$  the constant that holds zero or one, the FHN model can be given the form:

$$(4.2) \quad \begin{cases} u_t - \varepsilon u_{xx} + u + v = \bar{\eta} \\ v_t + \beta v - b u = 0 \end{cases}, \quad (x, t) \in \mathfrak{R}$$

and estimate (3.9) can be applied.

*Non linear case* - Consider now the non linear case defined by (1.2). By means of the previous results we are able to obtain integral equations for the two components  $(u, v)$  in terms of the data. All this implies, also in this case, a qualitative analysis of the solution together with a priori estimates.

At first let us observe that by (1.1)<sub>2</sub> one has:

$$(4.3) \quad v = v_0 e^{-\beta t} + b \int_0^t e^{-\beta(t-\tau)} u(x, \tau) d\tau$$

and this formula, together with (1.4) require the presence of the following convolutions:

$$(4.4) \quad K_i(r, t) = \int_0^t e^{-\beta(t-\tau)} K_{i-1}(x, \tau) d\tau \quad (i = 1, 2)$$

which explicitly are given by [21]:

$$(4.5) \quad K_i = \int_0^t \frac{e^{-\frac{x^2}{4\varepsilon y} - a y - \beta(t-y)}}{2\sqrt{\pi\varepsilon y}} \left( \sqrt{\frac{t-y}{by}} \right)^{i-1} J_{i-1}(2\sqrt{by(t-y)}) dy \quad (i = 1, 2).$$

As consequence, together with  $\theta_0$  defined by (3.6), the other two  $\theta$  functions

$$(4.6) \quad \theta_i(x, t) = \sum_{n=-\infty}^{\infty} K_i(x + 2nL, t) \quad (i = 1, 2)$$

must be considered.

To allow a plainer reading let's set



$$(4.7) \quad G_i(x, \xi, t) = \theta_i(|x - \xi|, t) + \theta_i(x + \xi, t) \quad (i = 0, 1, 2)$$

In this manner, owing to (3.7) one has:

$$(4.8) \quad \begin{aligned} u(x, t) = & \int_0^L [G_0(x, \xi, t) u_0(\xi) - G_1(x, \xi, t) v_0(\xi)] d\xi + \\ & - 2\varepsilon \int_0^t \theta_0(x, t - \tau) \psi_1(\tau) d\tau + 2\varepsilon \int_0^t \theta_0(L - x, t - \tau) \psi_2(\tau) d\tau \\ & + \int_0^t d\tau \int_0^L G_0(x, \xi, t - \tau) \varphi[\xi, \tau, u(\xi, \tau)] d\xi. \end{aligned}$$

As for the  $v$  component, by (4.3) one deduces:

$$(4.9) \quad \begin{aligned} v(x, t) = & v_0 e^{-\beta t} + b \int_0^L [G_1(x, \xi, t) u_0(\xi) - G_2(x, \xi, t) v_0(\xi)] d\xi + \\ & - 2b\varepsilon \int_0^t \theta_1(x, t - \tau) \psi_1(\tau) d\tau + 2b\varepsilon \int_0^t \theta_1(L - x, t - \tau) \psi_2(\tau) d\tau \\ & + b \int_0^t d\tau \int_0^L G_1(x, \xi, t - \tau) \varphi[\xi, \tau, u(\xi, \tau)] d\xi. \end{aligned}$$

Let us observe that the kernels  $K_1(x, t)$  and  $K_2(x, t)$  have the same properties of  $K_0(x, t)$ . In fact [21]:

**Teorema 4.1.** *For all the positive constants  $a, b\varepsilon, \beta$  it results:*

$$(4.10) \quad \int_{\mathfrak{R}} |K_1| d\xi \leq E(t); \quad \int_0^t d\tau \int_{\mathfrak{R}} |K_1| d\xi \leq \beta_1$$

$$(4.11) \quad \int_{\mathfrak{R}} |K_2(x - \xi, t)| d\xi \leq \int_0^t e^{-ay - \beta(t-y)} (t - y) dy \leq t E(t)$$

where  $E(t)$  is defined in (3.3)<sub>1</sub> and  $\beta_1 = (a\beta)^{-1}$ . ■

Now, let  $\|z\| = \sup_{\Omega_T} |z(x, t)|$ , and let  $\mathcal{B}_T$  denote the Banach space

$$(4.12) \quad \mathcal{B}_T \equiv \{z(x, t) : z \in C(\Omega_T), \|z\| < \infty\}.$$

By means of standard methods related to integral equations and owing to basic properties of  $K_i$ ,  $G_i$  ( $i = 0, 1, 2$ ) and  $\varphi(u)$ , it is easy to prove that the mapping defined by (4.8) [(4.9)] is a contraction of  $\mathcal{B}_T$  in  $\mathcal{B}_T$  and so it admits a unique fixed point  $u(x, t) \in \mathcal{B}_T$  [25, 26]. Hence

**Teorema 4.2.** *When the initial data  $(u_0, v_0)$  are continuous functions, then the Neumann [Dirichlet] problem related to the non linear (FHN) system (1.1), (1.2) has a unique solution in the space of solutions which are regular in  $\Omega_T$ . ■*

Continuous dependence for the solution of  $(P_N)$  [ $(P_D)$ ] is an obvious consequence of the previous estimates.

As for asymptotic properties, it is well known that the (FHN) system admits arbitrary large invariant rectangles  $\Sigma$  containing  $(0, 0)$  so that the solution  $(u, v)$ , for all times  $t > 0$ , lies in the interior of  $\Sigma$  when the initial data  $(u_0, v_0)$  belong to  $\Sigma$ . [?] For this, considering, for example, the case  $\psi_1 = \psi_2 = 0$  and letting

$$\|\varphi\| = \sup_{\Omega_T} |\varphi(x, t, u)|,$$

then by means of (4.8), (4.9) and owing to the estimates (4.12), (??), (4.10), (4.11), the following theorem can be stated:

**Teorema 4.3.** *For regular solution  $(u, v)$  of the (FHN) model, when  $\psi_1 = \psi_2 = 0$ , the following estimates hold:*

$$(4.13) \quad \begin{cases} |u| \leq 2[\|u_0\| (1 + \pi\sqrt{b}t) e^{-\omega t} + \|v_0\| E(t) + \beta_0 \|\varphi\|] \\ |v| \leq \|v_0\| e^{-\beta t} + 2[b(\|u_0\| + t\|v_0\|) E(t) + b\beta_1 \|\varphi\|] \end{cases}$$

■

Therefore, when  $t$  is large, the effect due to the initial disturbances  $(u_0, v_0)$  is exponentially vanishing while the effect of the non linear source is bounded for all  $t$ .

Asymptotic effects of boundary perturbations can be found in [6].

All the previous results can be applied to the boundary Dirichlet, too.

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